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WORD PROBLEMS FOR SEMIGROUPS *

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Duncan and Gilman introduced the concept “word problems” for finitely generated semigroups in terms of formal languages. In this paper, we show that finitely generated semigroups with a regular [resp. context-free] word problem has a presentation with regular [resp. context-free] congruence classes.

1. Word problems and semigroups with a presentation with special congruence classes

We recall regular, context-free and one counter languages (see [4]).

Definition 1 . Let X be finite alphabets. X^* is the set of all words over X . X^+ is the set of all non-empty words over X . A subset of X^* is called a **language** over X

Definition 2 . An automaton is defined as a 5-tuple:

$M = (Q, X, \delta, q_0, F)$ where Q is a finite set of states, q_0 is the start state, X is a finite set of the input alphabets, $\delta : Q \times X_\epsilon \rightarrow Q$ is the transition function, $X_\epsilon = X \cup \{\epsilon\}$, $F \subset Q$ is the set of accepting states.

For q_0 and $w = x_1x_2 \cdots x_{n-1}x_n (\in X^*)$, let $q = \delta(\cdots \delta(\delta(q_0, x_1), x_2) \cdots), x_n)$. Then we define $\delta(q_0, w) = q$.

The accepted language by the M

$$L(M) = \{w \in \Sigma \mid \delta(q_0, w) = q (q \in F)\}$$

is called a regular language.

*This is an abstract and the paper will appear elsewhere.

Definition 3 . A Pushdown automaton is defined as a 7-tuple:

$M = (Q, X, \Gamma, \delta, q_0, Z_0, F)$ where Q is a finite set of states, q_0 is the start state, X is a finite set of the input alphabets, Γ is a finite set of the stack alphabets, Z_0 is the initial stack symbol, $\delta : Q \times X_\epsilon \times \Gamma_\epsilon \rightarrow P(Q \times X_\epsilon)$ is the transition function, where $P(S)$ denotes the power set of S , $X_\epsilon = X \cup \{\epsilon\}$, $\Gamma_\epsilon = \Gamma \cup \{\epsilon\}$. $F \subset Q$ is the set of accepting states.

For $q \in Q$, $x \in X_\epsilon, w \in X^*$ and $Z \in \Gamma_\epsilon, \alpha \in \Gamma^*$, we define a relation \Rightarrow_M by $(q, xw, Z\alpha) \Rightarrow_M (p, w, \beta\alpha')$ if $(p, \beta) \in \delta((q, x, Z))$. The transitive closure of the relation \Rightarrow_M is written as a relation \Rightarrow_M^* . The accepted language by M

$$L(M) = \{w \in \Sigma \mid (q_0, w, Z_0) \Rightarrow_M^* (q, \epsilon, \epsilon) \ (q \in F)\}$$

is called a context-free language.

Definition 4 . A one counter machine is a pushdown automaton with a single stack alphabet. The accepted language by the one counter machine is called a one counter language.

Definition 5 . ([7]) A semigroup S has a presentation with regular [resp. context-free] congruence classes if there exists a finite set X and there exists a surjective homomorphism ϕ of X^+ to M such that for each word $w \in X^+$, $\phi^{-1}(\phi(w))$ is a regular [resp. context-free] language.

Let X be finite alphabets. For a word $w = x_1x_2 \cdots x_{n-1}x_n \in X^*$, $x_nx_{n-1} \cdots x_2x_1$ is called the reverse of w denoted by w^R .

Definition 6 . ([2]) Let S be a finitely generated semigroup, there exist finite alphabets X and there exists a surjective homomorphism ϕ of X^+ to S .

Then

(1) for each $w \in X^+$, the set $\{w\#v^R \mid \phi(w) = \phi(v)\}$ is called a word problem for the semigroup S .

(2) S has a regular [resp. regular context-free] word problem if the set $\{w\#v^R \mid \phi(w) = \phi(v)\}$ is a regular [resp. context-free] language over $X \cup \{\#\}$.

Theorem 1 . *A finitly generated semigroup with a regular [resp. context-free, one counter] word problem has a presentation with regular [resp. context-free, one counter] congruence classes.*

2. Regular word problems and semigroups with a presentation with regular congruence classes

The property of that a semigroup has a regular word problem is very strong. Actually, the following is known.

Result 1 . ([3], Proposition 3.1). *A semigroup S has a regular word problem if only if S is finite.*

On the other hand, we have

Definition 7 . *A semigroup S is called residually finite if for each pair of elements $s, t \in S$ there is a homomorphism ϕ of S such that $\phi(S)$ is a finite semigroup and $\phi(s)$ and $\phi(t)$ are not equal.*

Result 2 . ([7], Theorem 5) *If a finitely generated semigroup S has a presentation with regular congruence classes, then S is residually finite.*

Result 3 . ([3], Proposition 4.2) *If S is a finitely generated semigroup with a one counter word problem, then there exist finitely many elements $a_i, b_i, c_i \in S \cup \{\epsilon\}$ such that every element of S is represented by a word of the form $a_i b_i^n c_i$ for some i and some $n \geq 0$.*

3. Context-free word problems and semigroups with a presentation with context-free congruence classes

Definition 8 . *Let L be a language over A . The syntactic congruence σ_L on A^+ is defined by $w\sigma_L w'$ if and only if*

$$\{(x, y) \in A^+ \times A^+ \mid xwy \in L\} = \{(x, y) \in A^+ \times A^+ \mid xw'y \in L\}.$$

Then the factor semigroup A^+/σ_L is called syntactic semigroup of L denoted by $Syn(L)$.

Lemma 1 . ([7]). Let S be a finitely generated semigroup and ϕ a surjective homomorphism of A^+ to S .

For an element s of S , let $L = \phi^{-1}(s)$.

Define a congruence $\sigma_s = \{(a, b) \in S \times S \mid xay = s \text{ if and if } xby = s (x, y \in S^1)\}$ which is called a syntactic congruence on S .

Then the syntactic semigroup A^+/σ_L of L is isomorphic to the factor semigroup S/σ_s of S . (S/σ_s is called the syntactic semigroup of S at s .)

Theorem 2 . A semigroup S has a presentation with context-free [resp. one counter] congruence classes if and only if for each $s \in S$, the syntactic semigroup S/σ_s has context-free [resp. one counter] congruence classes.

Example 1 . Let $A = \{a, b\}$ and a context-free language $L = \{a^n b^n, b^n a^n \mid n \in \mathbb{N}\}$. Then all of σ_L -classes are $\overline{ab} = \{a^p b^p \mid p \in \mathbb{N}\}$, $\overline{a^n} = \{a^n\}$, $\overline{b^n} = \{b^n\}$, $\overline{a^{n+1}b} = \{a^{p+n} b^p \mid p \in \mathbb{N}\}$, $\overline{ab^{n+1}} = \{a^q b^{q+n} \mid q \in \mathbb{N}\}$, $\overline{ba} = \{b^p a^p \mid p \in \mathbb{N}\}$, $\overline{ba^{n+1}} = \{b^p a^{p+n} \mid p \in \mathbb{N}\}$, $\overline{b^{n+1}a} = \{b^{q+n} a^q \mid q \in \mathbb{N}\}$. $0 = A^* b A^* a A^* b A^* \cup A^* a A^* b A^* a A^*$. Then

- (1) L is a one counter language.
- (2) $\text{Syn}(L)$ has a linear growth function. Actually, $(|\overline{A^n}| \leq 6n)$.
- (3) $\text{Syn}(L)$ have a context-free word problem but does not have a one counter word problem.
- (4) $\text{Syn}(L)$ has a presentation with one counter congruence classes.

Example 2 . Let $A = \{a, b\}$ and $L = \{a^n b^m a^m b^n \mid m, n \in \mathbb{N}\}$. Then all of σ_L -classes are $\overline{a^i} = \{a^i\}$ ($i \in \mathbb{N}$), $\overline{b^i} = \{b^i\}$ ($i \in \mathbb{N}$), $\overline{a^i b^j} = \{a^i b^j\}$ ($i, j \in \mathbb{N}$), $\overline{b^i a^j} = \{b^i a^j\}$ ($i, j \in \mathbb{N}$), $\overline{a^i b^j a^k} = \{a^i b^j a^k\}$ ($i, j, k \in \mathbb{N}$), $\overline{b^i a^j b^k} = \{b^i a^j b^k\}$ ($i, j, k \in \mathbb{N}$), $\overline{a^i b^j a^j b} = \{a^i b^j a^j b\}$ ($i, j \in \mathbb{N}$), $\overline{ab^j a^j b^i} = \{ab^j a^j b^i\}$ ($i, j \in \mathbb{N}$). $0 = \cup_{i \neq j} A^* a b^i a^j b A^*$.

Then

- (1) L is a context-free language.
- (2) $\text{Syn}(L)$ does not have a presentation with context-free congruence classes.

Example 3 . Let $A = \{a, b, c\}$ and $L = \{a^n b^n c^n | n \in \mathbb{N}\}$. Then all of σ_L -classes are $\{1\}$, $\overline{a^q b^r} = \{a^q b^r\}$, $(q, r \in \mathbb{N})$, $\overline{b^q c^r} = \{b^q c^r\}$, $(q, r \in \mathbb{N})$, $\overline{a^i b^q c^r} = \{a^i b^q c^r | q, r, i \in \mathbb{N}, 2 \leq r \leq q\}$, $\overline{a^p b^q c} = \{a^p b^q c^i | p, q, i \in \mathbb{N}, 2 \leq p \leq q\}$, $\overline{abc} = \{a^n b^n c^n | n \in \mathbb{N}\}$, $0 = X^* b a X^* \cup X^* c b X^* \cup X^* c a X^* \cup X^* a c X^* \cup \{a^p b^q c^r | p, q, r \in \mathbb{N}, \text{either } q \leq p \text{ or } q \leq r\}$.
Then

(1) $\text{Syn}(L)$ has a polynomial growth function. Actually, $\left(|\overline{A^n}| = \frac{n(n+1)}{2}\right)$.

Therefore,

(2) $\text{Syn}(L)$ does not have a one-counter word problem.

(3) $\text{Syn}(L)$ has a presentation with one counter congruence classes.

Finally we pose the following two open problems.

Problem 1 . Characterize a finitely generated semigroup S with a one counter word problem.

Problem 2 . Characterize a finitely generated semigroup S which has a presentation with one counter congruence classes.

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